WAVE MOTIONS IN A VISCOUS FLUID LAYER IN THE PRESENCE OF SURFACTANTS

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Waves in viscous fluid layers covered by a layer of surfactants of arbitrary elasticity and bounded by a solid surface or just a gas are examined on the basis of the exact solution of the linearized Navier–Stokes equations. In the particular case of thin films, van der Waals forces are taken into account in addition to the capillary and gravitational forces. The influence of the layer thickness on the amplitude of the maximum damping decrement of the wave is determined. For thin films in a gas there exists a critical value of the surface elasticity for which the rate of perturbation growth is slowed down when it is exceeded. Far from the neighborhood of the critical value, the rate of growth of the perturbations depends weakly on the elasticity of the surface layer. The questions under consideration are important in investigations of the stability of liquid foams.

The hydrodynamic approach to the phenomenon of capillary-gravitational wave quenching in a fluid by surfactants has been developed by Levich [1]. Solutions have been given of the problems of wave damping on the surface of an infinitely deep fluid in the presence of surfactants of arbitrary concentration. The problem of wave damping in a fluid of finite depth has been solved in the particular case of an incompressible surfactant layer for high Reynolds number in [2]. Recently, a large number of experimental papers on the quenching of capillary waves has appeared (see [3], as well as the brief survey in [4]). It has been disclosed that, for definite values of the elasticity of the surfactant layer, the wave damping exceeds the damping corresponding to infinite elasticity.

1. Fluid of Finite Depth. Fundamental Equations. A plane-parallel layer of viscous incompressible fluid with surface tension σ bounds a gas, at least on one side, and its free surface is covered by a surfactant layer. The Navier–Stokes equations for low-amplitude waves are written in the linearized form

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p + \mathbf{F} + \mu \Delta \mathbf{v}, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

If the fluid layer bounds a solid surface, then the gravity force $F_z = \rho g$ is taken into account. (The z axis is directed toward the solid boundary, and g is the acceleration of gravity.) In the particular case of long waves in thin films, it is interesting to take account of the van der Waals forces of attraction acting in the fluid besides the capillary and gravitational waves. It is convenient to take account of these forces by including the volume force F_x in the component (the x axis is along the film); then the pressure p in the appropriate quantity will differ from the total pressure. However, the form of the equations (1.1) does not change. On the basis of known results (see [5-7], for example), it is possible to write for small perturbations of the layer thickness h

$$F_x = Q \partial h / \partial x, \qquad Q = A / 2\pi h^4 \tag{1.2}$$

The constant A in (1.2) equals the Hammacher constant ($A > 0$) for a film in a gas. In specific cases, $A < 0$ can hold for a layer on a solid surface.

The influence of surfactants is taken into account according to [1]. In the case of plane waves on a free surface, the tangential stress is

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$$p_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) = -\frac{\partial \sigma}{\partial x} \quad \text{for} \quad z = 0 \tag{1.3}$$

The equation of surfactant conservation should be added to (1.3), as should the expression for the elasticity of the surface layer ε. Neglecting surface diffusion,

$$\frac{\partial \Gamma}{\partial t} + \Gamma \frac{\partial v_x}{\partial x} = 0, \qquad \Gamma \frac{d\sigma}{d\Gamma} = \varepsilon \quad (\varepsilon < 0) \tag{1.4}$$

can be written [1] for the surface concentration Γ of insoluble surfactants.

For small displacements ζ of the fluid particles along the z axis, the pressure p_0 in the gas and the pressure p near the free surface differ by

$$p - p_0 = 2\mu \frac{\partial v_z}{\partial z} + \sigma \frac{\partial^2 \zeta}{\partial x^2} \quad \text{for} \quad z = 0 \tag{1.5}$$

The kinematic conditions on the free surface and on the solid boundary are

$$v_z = \partial \zeta / \partial t \quad \text{for} \ z = 0, \qquad v_x = v_z = 0 \quad \text{for:} \ z = h \tag{1.6}$$

For the case of a fluid of finite depth, the general wave solution of (1.1) can be written as follows:

$$\begin{aligned} v_x &= i\,(k\,(A_1 \operatorname{ch} kz + A_2 \operatorname{sh} kz) + l'\,(B_1 \operatorname{ch} l'z + B_2 \operatorname{sh} l'z))\, e^{ikx+\alpha' t} \\ v_z &= k\,(A_1 \operatorname{sh} kz + A_2 \operatorname{ch} kz + B_1 \operatorname{sh} l'z + B_2 \operatorname{ch} l'z)\, e^{ikx+\alpha' t} \\ p &= p_0 + \rho g z - Q\zeta(0) - \rho\alpha'\,(A_1 \operatorname{ch} kz + A_2 \operatorname{sh} kz)\, e^{ikx+\alpha' t} \\ &\quad (l'^2 = \alpha'/\nu + k^2) \end{aligned} \tag{1.7}$$

The pressure p has here been found by taking account of (1.2), where $\zeta(0) = h_0 - h$ (h_0 is the thickness of the unperturbed layer). The coefficients in (1.7) can be found from (1.3)-(1.6). It is natural to seek the displacement in the surface $\zeta(0)$ and the surface concentration Γ as

$$\zeta(0) = \zeta_1 e^{ikx+\alpha' t}, \qquad \Gamma = \Gamma_0 + \Gamma_1 e^{ikx+\alpha' t} \tag{1.8}$$

To simplify the computations, it is convenient to introduce the notation

$$\alpha = \alpha'/\nu k^2, \quad \Omega = (k^{-2}(\rho g - Q) + \sigma)/\rho\nu^2 k, \quad \delta = -\varepsilon/\rho\nu^2 k, \quad l = \sqrt{\alpha + 1}, \quad a = kh \tag{1.9}$$

Here, the branch of the function $\sqrt{z}$ in the complex plane has been chosen by using a slit from 0 to $-\infty$ along the real axis. After the elimination of Γ_1 the following system of equations can be obtained from (1.3)-(1.9):

$$\begin{aligned} (2+\alpha) A_1 + 2lB_1 &= (A_2 + B_2)\,\Omega/\alpha \\ 2A_2 + (2+\alpha) B_2 &= (A_1 + lB_1)\,\delta/\alpha \\ A_1 \operatorname{ch} a + A_2 \operatorname{sh} a + lB_1 \operatorname{ch} la + lB_2 \operatorname{sh} la &= 0 \\ A_1 \operatorname{sh} a + A_2 \operatorname{ch} a + B_1 \operatorname{sh} la + B_2 \operatorname{ch} la &= 0 \end{aligned} \tag{1.10}$$

A complex conjugate corresponds to every solution of the system (1.10), which is an expression of the invariance of the problem relative to a change in direction along the x axis. An equation for α follows from the condition of compatibility of (1.10). The imaginary part of α determines the frequency of oscillation, and the real part the damping decrement. The value of α can be found analytically in the limit cases $|\alpha| a^2 \ll 1$ and $|\alpha| a^2 \gg 1$ as well as within the limit of high ν ($|\alpha| \ll 1$) or low ($|\alpha| \gg 1$).

2. Waves for $|\alpha| a^2 \ll 1$. Low Viscosity. The unknowns B_1 and B_2 in the system (1.10) can be replaced as follows in the case $|l|a \gg 1$:

$$B_1 = B, \qquad B_2 = f e^{-la} - B \tag{2.1}$$

Neglecting the exponentially small terms exp (−Re la) in (1.10) after having eliminated f yields the following equations:

$$\begin{aligned} (2+\alpha) A_1 + 2lB = (A_2 - B)\,\Omega/\alpha, \quad 2A_2 - (2+\alpha) B = (A_1 + lB)\,\delta/\alpha \\ A_1 (1 - l \operatorname{th} a) + A_2 (\operatorname{th} a - l) = 0 \end{aligned} \tag{2.2}$$

The compatibility condition of the system (2.2) allows us to obtain:

$$\alpha^2((2+\alpha)^2 - \varphi\Omega + 4l\varphi) + \delta(l\alpha^2 - l\varphi\Omega - \Omega) = 0$$
$$\varphi = (1 - l\,\mathrm{th}a)(l - \mathrm{th}a)^{-1} \tag{2.3}$$

The solution of (2.3) for $\Omega \gg 1$, $\Omega a^5 \gg 1$ can be found by iteration. Two approximations are sufficient for a qualitative investigation. Taking the initial point $\alpha = \alpha_1$ corresponding to the value in an ideal fluid, just as in [1], the following asymptotic formulas can be obtained from (2.3):

$$\alpha = \alpha_1 + \alpha_2, \qquad \alpha_1 = i\sqrt{\Omega\,\mathrm{th}\,a}$$
$$\alpha_2 = -2 + (1 + (1 + \mathrm{ch}^2 a)\,\delta\alpha_1^{-3/2})\,\Omega\,/\,2\,\mathrm{ch}^2 a\,(\alpha_1^{5/2} + \delta) \tag{2.4}$$

Here and henceforth, only one root of the conjugate pairs will be written down. As is seen from (1.7) and (2.2), the solution (2.4) corresponds to a wave in which the viscosity is manifest only in a thin layer near the surface [8], while the motion in the volume is similar to ideal fluid motion. A solution of (2.3) exists, in addition to (2.4), which is for $\delta \gg 1$, $\delta a^3 \gg 1$

$$\alpha_1 = {}^1/_2(i\sqrt{3} - 1)\,\delta^{2/3}, \qquad \alpha_2 = {}^2/_3(\Omega\,/\,\delta - {}^7/_2)\,(1 - \Omega\,\mathrm{th}\,a\,/\,\delta\sqrt{\alpha_1})^{-1}.$$

According to (1.7) and (2.2), wave motion in a thin layer of thickness $\sim|l'|^{-1}$ near the free surface corresponds to this latter solution, where $|v_x| \gg |v_z|$. This "near-surface" wave appears only because of the presence of surfactants; its damping is high and independent of the layer thickness in a first approximation. Formulas for the frequency shift ξ and damping ζ of the "volume" wave follow from (2.4)

$$\xi = \mathrm{Im}\,\alpha_2 = -\Omega\,[\mathrm{ch}^{-2} a - 2y\,(1 + y^2)^{-1}]/2\delta_m$$
$$\zeta = \mathrm{Re}\,\alpha_2 = -2 - \Omega\,[\mathrm{ch}^{-2} a + 2\,(1 + y^2)^{-1}]/2\delta_m \tag{2.5}$$
$$y = \delta_m/\delta - 1, \qquad \delta_m = \sqrt{2}\,(\Omega\,\mathrm{th}\,a)^{3/4}$$

The passage to the limit as $a \to \infty$ in the last formulas yields appropriate formulas of the theory of capillary-gravitational wave damping in a fluid of infinite depth [1] (to the accuracy of the notation). If we pass to the limit as $\delta \to \infty$ (an incompressible surfactant layer), and neglect the first members in the formula for ζ (2.5), the formulas then yield the result in [2], which has been obtained by another method. A negative frequency shift has been noted in [2]. According to (2.5), the frequency shift is a nonmonotonic function of the elasticity δ, negative for $\delta = 0$ and $\delta \to \infty$, positive for

$$\delta_m\,(1 + \mathrm{ch}^2 a + \sqrt{\mathrm{ch}^4 a - 1})^{-1} < \delta < \delta_m\,(1 + \mathrm{ch}^2 a - \sqrt{\mathrm{ch}^4 a - 1})^{-1}$$

As the layer thickness diminishes ($a \to 0$) the domain of the positive frequency shift degenerates into the point $\delta_k = (\Omega\,\mathrm{th}a)^{3/4}/\sqrt{2}$. For any a the damping ζ reaches the value of the damping for an incompressible layer ζ_∞ for $\delta = \delta_k$. For $\delta > \delta_k$ the wave damping is greater than in the case of an incompressible layer. The maximum value of the damping equals

$$\zeta_m = \zeta_\infty\,(1 + 2\,\mathrm{ch}^2\,a)\,(1 + \mathrm{ch}^2\,a)^{-1} \quad \text{for} \quad \delta = \delta_m \tag{2.6}$$

It is seen from the expression for δ_k and (2.6) that for long waves ($a \ll 1$) about $a^{-3/4}$ less elasticity of the surfactant layer is required than in the case of a fluid of infinite depth in order to achieve the quenching effect of the incompressible surfactant layer, i.e., free-surface stabilization is achieved more easily in the layer than in the depth of the fluid.

The dependence of the frequency shift $f = \xi$ and damping $f = \zeta$ on the dimensionless elasticity coefficient

$$\delta^\circ = \frac{\delta}{\sqrt{2}\,(\Omega\,\mathrm{th}\,a)^{3/4}}$$

is presented in Figure 1. Curves 1 and 3 describe the limit cases $a \to \infty$, $a \to 0$, and curve 2 is for $a = 1$. As the layer thickness diminishes, the amplitude of the damping maximum referred to the damping for an incompressible layer ζ_∞ is reduced from 2 to 1.5, and the domain of elasticity values in which the frequency shift is positive vanishes in the limit $a \to 0$ ($a \gg |l|^{-1}$). It is interesting to clarify the nature of the

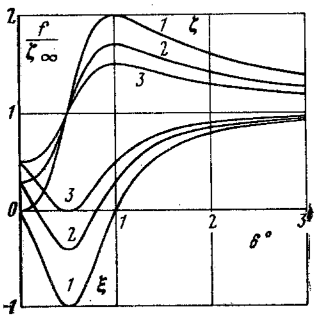

Fig. 1

wave motion at the maximum point. From (1.7), (2.1), (2.2), and (2.4) it is possible to obtain in the limit $|l|a \gg 1$, $|l| \gg 1$ for the ratio between the velocities at the surface

$$v_z / v_x = \operatorname{th} a\,(i + (1 - i)\,\delta / \sqrt{2}\,(\Omega \operatorname{th} a)^{3/4}) \tag{2.7}$$

It is seen that the fluid particle trajectories on the surface are ellipses. A rise in elasticity δ results in rotation of the principal axes of the ellipse and a change in the eccentricity. As is seen from (2.7), the ellipse degenerates into a segment at the point of the maximum (2.6). During wave propagation the fluid particles near the free surface perform oscillations along lines at an angle $\tan^{-1}(\tanh a)$ to the layer surface. The particle trajectories will be ellipses as they recede from the free surface, where the trajectory shape will be the same as in an ideal fluid for $z \gg |\mathrm{Re}\, l'|^{-1}$.

3. Low Reynolds Numbers. Long waves ($a \ll 1$) are considered in a fluid layer on a solid surface for $|l|a \ll 1$, which is possible in the domain $|\Omega|a^5 \ll 1$. The last two equations in (1.10) yield in this limit case

$$a\alpha B_2 = -(1 + {}^1/_{12}a^4)A_1 - (1 + {}^1/_2\alpha a^2 + {}^1/_{12}l^2a^4)\,lB_1$$
$$A_2 + B_2 = -{}^1/_2(A_1 + lB_1)\,a + {}^1/_{12}(A_1 + l^3B_1)\,a^3$$

Eliminating A_2 and B_2 from the first two equations in (1.10) by using these formulas, the following equation:

$$\alpha(\alpha + {}^1/_3\Omega a^3) + (\alpha + {}^1/_{12}\Omega a^3)\,a\delta = 0 \tag{3.1}$$

can be obtained in the limit $a \ll 1$, $|l|a \ll 1$.

It follows from (3.1) that there are no oscillating motions. Let $\Omega > 0$; then the roots of (3.1) are nonpositive.

The root α_1, which equals $-{}^1\!/_3\Omega a^3$ for $\delta = 0$ (pure surface), diminishes monotonically and $\alpha_1 \to -\infty$ as $\delta \to \infty$; i.e., the relaxation time tends to zero. For sufficiently large δ, the wave motion emerges from the domain of low Reynolds numbers under consideration.

The solution α_2, equal to zero for $\delta = 0$, decreases monotonically, $\alpha_2 \to -{}^1\!/_{12}\Omega a^3$ as $\delta \to \infty$. The large values of the relaxation time for wave motion corresponding to α_2 are explained for small δ by the magnitude of the characteristic force exciting this flow.

Let van der Waals forces play a decisive role in the film; then $\Omega < 0$ according to (1.2) and (1.9). One root of (3.1) is hence positive. The solution α_1, corresponding to a wave of growing amplitude, varies monotonically between $-{}^1\!/_3\Omega a^3$ and $-{}^1\!/_{12}\Omega a^3$. The passage to the limit case of an incompressible surfactant layer occurs gradually, where the layer is incompressible if $\delta \gg {}^1\!/_3|\Omega|a^2$. For long waves this latter condition can be considerably weaker than the condition $\delta > \sqrt{2}(\Omega a)^{3/4}$, which is valid in the low viscosity domain. Therefore, surfactants act especially effectively on long waves in thin layers.

4. Fluid Film in a Gas. A liquid layer in a gas is examined, where the free surfaces are covered by identical surfactant layers. Let the origin be in the plane of symmetry of the film, and let the z axis be perpendicular thereto. A symmetric solution (the pressure is an even function of z) of the Navier–Stokes equations (1.1) can be extracted from (1.7)

$$\begin{gathered} v_x^+ = i\,(kA_1 \operatorname{ch} kz + l'B_1 \operatorname{ch} l'z)\,e^{ikx+\alpha' t} \\ v_z^+ = k\,(A_1 \operatorname{sh} kz + B_1 \operatorname{sh} l'z)\,e^{ikx+\alpha' t} \\ p = p_0 - 2Q\zeta(-h/2) - \rho\alpha' A_1 \operatorname{ch} kz\, e^{ikx+\alpha' t} \end{gathered} \tag{4.1}$$

Here (1.2) has been applied in calculating p and it has been taken into account that $h - h_0 = \zeta(\tfrac{1}{2}h) - \zeta(-\tfrac{1}{2}h)$. By virtue of the symmetry it is sufficient to satisfy the boundary conditions for $z = -\tfrac{1}{2}h$. Equations for the coefficients in (4.1) can be obtained from (1.3)–(1.5) and the first equation of (1.6) written for $z = -\tfrac{1}{2}h$. For simplification, the following notation is introduced:

$$a = {}^1/_2 kh, \qquad A_+ = A_1 \operatorname{ch} a, \qquad B_+ = B_1 \operatorname{ch} ({}^1/_2 l'h) \tag{4.2}$$

Also,using all the notation (1.9) in addition to this latter, it is possible to find, finally,

$$\begin{gathered}(2+\alpha) A_+ + 2lB_+ = -(A_+ \operatorname{th} a + B_+ \operatorname{th} la)\,\Omega / \alpha \\ 2A_+ \operatorname{th} a + (2+\alpha) B_+ \operatorname{th} la = -(A_+ + lB_+)\,\delta / \alpha \\ (\Omega = (\sigma - 2Qk^{-2})/\rho v^2 k, \quad l = \sqrt{\alpha + 1})\end{gathered} \tag{4.3}$$

Hence, the equation for α follows:

$$(\Omega \operatorname{th} a + (2+\alpha)^2) \operatorname{th} la - 4l \operatorname{th} a + \delta\,(l - \Omega\,(\operatorname{th} la - l \operatorname{th} a)\,\alpha^{-2}) = 0 \tag{4.4}$$

Antisymmetric waves determined from (1.1) by analogy with (1.7) in the form

$$\begin{gathered} v_x^- = i\,(kA_2 \operatorname{sh} kz + l'B_2 \operatorname{sh} l'z)\, e^{ikx + \alpha' t} \\ v_z^- = k\,(A_2 \operatorname{ch} kz + B_2 \operatorname{ch} l'z)\, e^{ikx+\alpha' t} \\ p = p_0 - \rho\alpha' A_2 \operatorname{sh} kz\, e^{ikx+\alpha' t}\end{gathered} \tag{4.5}$$

enter in addition to the symmetric waves in a complete system of waves.

It is important that the forces taken into account in (1.2) should not enter here since $v_z^- \zeta(\frac{1}{2}h) = \zeta(-\frac{1}{2}h)$ because of evenness. Van der Waals forces do not affect the antisymmetric waves. Using the notation

$$a = {}^1/_2 kh, \qquad A_2 \operatorname{sh} ({}^1/_2 kh) = A_-, \qquad B_2 \operatorname{sh} ({}^1/_2 l'h) = B_- \tag{4.6}$$

the following equations can be obtained from (1.3)-(1.5) and the first equation in (1.6) for $z = -\frac{1}{2}h$:

$$\begin{gathered}(2+\alpha) A_- + 2lB_- = -(A_- \operatorname{cth} a + B_- \operatorname{cth} la)\,\Omega'/\alpha \\ 2A_- \operatorname{cth} a + (2+\alpha) B_- \operatorname{cth} la = -(A_- + lB_-)\,\delta/\alpha \\ (\Omega' = \sigma / \rho v^2 k, \quad l = \sqrt{\alpha+1})\end{gathered} \tag{4.7}$$

It should be noted that the form of writing the boundary conditions at $z = \frac{1}{2}h$ differs in sign from (1.3) and (1.5). It can be shown that because of the symmetry of the problem these conditions will be satisfied automatically for symmetric and antisymmetric waves if the corresponding conditions are satisfied for $z = -\frac{1}{2}h$. The equation for α in the case of antisymmetric waves should follow from (4.7):

$$[\Omega' \operatorname{cth} a + (2+\alpha)^2] \operatorname{cth} la - 4l \operatorname{cth} a + \delta\,[l - \Omega'\,(\operatorname{cth} la - l \operatorname{cth} a)\,\alpha^{-2}] = 0 \tag{4.8}$$

The equations for the different kinds of waves (4.4) and (4.8) differ by the replacement $\operatorname{th} a \to \operatorname{cth} a$, $\operatorname{th} la \to \operatorname{cth} la$, $\Omega \to \Omega'$.

The complex conjugate corresponds to each root of these equations. On going to a layer of large thickness ($a \to \infty$) the difference between (4.4) and (4.8) vanishes, where both these equations also agree with (2.3) in the limit. Equations (4.4) and (4.8) admit of analytical investigation in the limit cases $|\alpha| a^2 \gg 1$ and $|\alpha| a^2 \ll 1$.

5. Low Viscosity. For $|l| a \gg 1$, $|l| \gg 1$, equations (4.4) and (4.8) can be solved by iteration. Exactly as for a film on a solid surface, waves are possible here which are localized near the free surface in a thin layer $\Delta z \sim |l'|^{-1}$, as are waves associated with motion in a volume. Below, the "volume" waves are examined. The following asymptotic expressions for the frequency $\operatorname{Im} \alpha_1$, the frequency shift, and the damping are obtained from (4.4), analogously to (2.5), in the case of symmetric waves:

$$\begin{gathered}\alpha_1 = i\sqrt{\Omega q} \quad (\alpha = \alpha_1 + \alpha_2), \qquad \xi = \operatorname{Im} \alpha_2 = y\Omega^{1/4}/(1+y^2)\, q^{3/4}\sqrt{2} \\ \zeta = \operatorname{Re} \alpha_2 = -2 - \Omega^{1/4}/(1+y^2)\, q^{3/4}\sqrt{2}, \qquad y = \sqrt{2}\,(\Omega q)^{3/4}\delta^{-1} - 1\end{gathered} \tag{5.1}$$

Here $q = \tanh a$, $a = \frac{1}{2}kh$. It follows from (5.1) that there is a maximum damping for $\delta = \sqrt{2}(\Omega q)^{3/4}$. Wave damping in the case of an incompressible layer corresponds to the minimum. As the parameter a diminishes, symmetric wave damping increases, and the damping maximums are observed for all lesser values of the coefficient of elasticity ($\delta \sim a^{3/4}$). In contrast to a film on a solid substrate, where the diminution in thickness results in a reduction in the relative amplitude (ζ/ζ_∞) of the damping maximum, the relative amplitude of the maximum is not lowered here. As the film thickness diminishes, the approximation $|\alpha|a^2 \gg 1$ becomes invalid sufficiently rapidly. Formulas (5.1) remain valid for $\Omega \gg 1$, $\Omega a^5 \gg 1$. Somewhat different results are obtained in the case of antisymmetric waves. Expressions for the frequency, the frequency shift, and damping which agree with (5.1) can be derived for these waves from (4.8) if we put therein

$$q = \operatorname{cth} a, \quad \Omega = \sigma / \rho v^2 k \quad (a = kh/2)$$

It is seen from (5.1) that the damping of an antisymmetric wave decreases as the film thickness h diminishes, but the oscillation frequency grows. A damping maximum is observed for all large values of the elasticity of the surfactant layer, where its amplitude decreases as $h^{3/4}$. For small a antisymmetric waves are the bending oscillations of a thin layer of viscous fluid enclosed between two stretched elastic membranes. In the case of antisymmetric waves, the approximation $|\alpha|a^2 \gg 1$ is applicable in a larger range of values of a than in the case of symmetric waves. Formulas (5.1) are valid in the case of antisymmetric waves ($q = \coth a$) for $\Omega' a^3 \gg 1$, $\Omega' \ll 1$.

6. Antisymmetric Waves for $|\alpha|a^2 \ll 1$. If $\Omega' a^3 \ll 1$, then it can be considered that $|l|a \ll 1$ in (4.8), and by using the expansion of coth x at zero, the following equation, which is valid in the domain $|\alpha| \gg 1$, $|\alpha|a^2 \ll 1$, can be obtained:

$$\alpha (\Omega'/a + \alpha^2) + \delta (a\alpha (1 + \alpha) + \Omega') = 0 \tag{6.1}$$

If the elasticity δ is large, then purely damped motions are possible. For $\delta \gg (\Omega'/a^3)^{1/2}$ one of the roots of (6.1) will be $\alpha \approx -a\delta$. In the case of a pure surface there follows from (6.1)

$$\alpha_1 = i\sqrt{\Omega'/a} \qquad (\alpha' = i\sqrt{2\sigma k^2/\rho h})$$

The damping of oscillations corresponding to this root is zero in a first approximation. For an incompressible layer ($\delta \to \infty$) the frequency differs slightly from Im α_1, and the dimensionless damping decrement equals $1/2$. As is seen from (6.1), the passage to an incompressible layer occurs for $\delta \sim (\Omega'/a^3)^{1/2}$.

Therefore, as the film thickness diminishes, the influence of surfactants on antisymmetric waves decreases. For small a an elasticity δ considerably exceeding the elasticity $\delta \sim \Omega a^2$, say, which is needed to stabilize the free surface of a film on a solid substrate, is required for complete stabilization of the surface relative to a wave of the given length.

Nevertheless, it is impossible to consider the influence of surfactants on damping of waves of the type under consideration as weak. The fact is that taking account of the second approximation in (4.8) yields Re $\alpha = -2/3 a^2$ in the case of a pure surface. Hence, for long waves ($a \ll 1$) damping because of surfactants can grow about a^{-2}-fold. Under real conditions, the quantity a can be on the order of 10^{-3} and less, which would result in practically undamped waves in the case of a pure surface.

The presence of a gas medium surrounding the film can be a factor, in addition to the surfactants, which will considerably increase the damping of the waves under consideration. Taking account of the gas motion during oscillations of the film results in the following. The magnitude of the damping in the case of a pure surface is independent of the motion in the gas for $a^3(a/\Omega')^{1/4} \gg \mu'/\mu$ (μ' is the dynamic gas viscosity). The gas motion can be neglected in the calculation of oscillation frequency only for $a \gg \rho'/\rho$ (ρ' is the gas density). If $\mu \sim 10^{-2}$ g/cm · sec and the film is in air, then for $a \lessdot 10^{-3}$ the frequency of waves of this type will depend primarily on the inertial properties of the gas, and as the film thickness diminishes further the growth in oscillation frequency will cease.

7. Symmetric Waves for $|\alpha|a^2 \ll 1$. Neglecting small quantities, (4.4) in the limit $|l|a \ll 1$ can be rewritten as follows:

$$a\alpha (\Omega a + 4\alpha + \alpha^2) + (\alpha + 1/3\, \Omega a^3)\, \delta = 0 \tag{7.1}$$
$$(\Omega = (\sigma - 2Qk^{-2}) / \rho v^2 k)$$

If capillary forces play the main role in the film, then $\Omega > 0$. It is seen from (7.1) that if $\Omega a \gg 1$, then the wave motion is oscillatory for $\delta = 0$, and there is also aperiodic motion with the damping decrement $1/3 \Omega a^3$ for $\delta \to \infty$. For $\Omega a^5 \ll 1$, the following approximate formulas can be found for the roots of (7.1):

$$\alpha_{1,3} = -2 \pm \sqrt{4 - a^{-1}\delta - \Omega a}, \qquad \alpha_2 = -\frac{a\delta}{3(1 + \delta/\Omega a^2)} \tag{7.2}$$

If $\Omega a \gg 1$, then for $\delta \ll \Omega a^2$ there are waves caused by capillary forces. Two traveling waves, differing in the propagation direction, correspond to the roots α_1 and α_3. For $\delta \gg \Omega a^2$ the surface tension of the liquid does not influence the waves corresponding to α_1 and α_3. Elasticity of the surfactant surface layer is exclusively the reason for the appearance of these waves. The velocity v_x hence varies insignificantly along the film cross section, as is seen from (4.1). The fact that some mass of fluid in each film section seems to be "glued" to two elastic "membranes" because of viscosity is the physical reason for the origin of such waves. The dimensionless frequency equals $\sim\sqrt{\delta/a}$, and the damping decrement of about 2 is relatively small. Waves of this kind are described correctly by (7.1) while $\delta a^3 \ll 1$.

Waves caused by the elasticity of the surface layer have been examined in [9, 10], where (7.1) and the first formula in (7.2), in particular, have been obtained in a somewhat different form.*

The second formula in (7.2) is valid for any δ. Relative to the aperiodic motion corresponding to α_2, the surfactant layer behaves as though incompressible for $\delta \gg \Omega a^2$; the characteristic relaxation time hence increases approximately a^{-2}-fold under the influence of surfactants.

The quantity Ω in (7.1) can be negative because of van der Waals forces. This is possible for long waves in thin films since the Hammacher constant A in (1.2) is on the order of 10^{-12} erg (see [7], for example). In the case $\Omega < 0$, it is convenient to rewrite (7.1) in a new notation

$$\delta_* = -\Omega a^2, \qquad \delta = \delta_*(1 + as), \quad \alpha = z\delta_* \qquad (7.3)$$
$$\delta_* z^3 + 4z^2 + sz - {}^1/_3(1 + as) = 0$$

Let $\delta_* \ll 1$. Then applying the method of asymptotic coalescence to the solution of (7.3), the following asymptotic formulas can be found for the roots:

$$z_1 = (\sqrt{1 - {}^1/_4\delta_* s} - 1)\,2/\delta_*, \quad z_2 = {}^1/_3(a + s^{-1}) \quad \text{for} \quad s \gg 1$$
$$z_1 = -{}^1/_8(s + \sqrt{s^2 + {}^{16}/_3}) \quad z_2 = {}^1/_8(-s + \sqrt{s^2 + {}^{16}/_3}) \quad \text{for} \quad |s| \ll M \qquad (7.4)$$
$$z_1 = {}^1/_3(a + s^{-1}), \quad z_2 = (\sqrt{1 - {}^1/_4\delta_* s} - 1)\,2/\delta_* \quad \text{for} \quad s \ll -1$$
$$z_3 = -(\sqrt{1 - {}^1/_4\delta_* s} + 1)\,2/\delta_* \quad (M = \min(a^{-1}, \delta_*^{-1}))$$

For $\delta_* \gtrsim 1$ formulas (7.4) will be suitable in the domain $|s| \gg \delta_*^{1/3}$. Values of α corresponding to z_j can be denoted by α_j (j = 1, 2, 3). As is seen from (7.4), the root α_2 corresponds to a wave of growing amplitude. The roots α_1 and α_3 correspond either to aperiodic motions (for $\delta - \delta_* < 4a$), or to traveling waves with frequency $\sqrt{(\delta - \delta_*)/4a - 1}$ and damping decrement two (dimensionless notation).

It follows from (7.3) and (7.4) that as δ passes through the circle δ_* a sharp change in α_1 and α_2 occurs.

The inequality $|\Omega|a \gg 1$ is the condition for a slight change in δ_*. For $|\Omega|a \lesssim 1$ the quantity α varies substantially over the whole domain $0 < \delta < \delta_*$. If $|\Omega|a \gg 1$, then $\delta_* \gg a$. Hence, as δ increases in the neighborhood of δ_*, the quantity α_2 diminishes approximately $(a\delta_*)^{-1}$-fold, and $|\alpha_1|$ increases the same number of times.

In films existing in a foam, $(a\delta_*)^{-1}$ can be a quantity on the order of tens of thousands and greater. Hence, although the behavior of α_2 is described by smooth functions according to (7.4), it can be considered in practice that the film is not stabilized for $\delta < \delta_*$, and is stabilized for $\delta > \delta_*$. This deduction is in agreement with the results of the authors of [9, 10], obtained by numerical computations on an electronic computer.

If the capillary forces in a wave of growing amplitude are small as compared with the van der Waals forces $k^2\sigma \ll A/\pi h^4$, then $|\varepsilon| > {}^1\!/_4 A/\pi h^2$ will be the condition for a small difference in the surfactant from an incompressible layer on the basis of (1.2), (4.3), (7.3), and (7.4). For lesser elasticities $|\varepsilon|$, the film should rupture practically instantaneously. It can also be deduced from (7.3) and (7.4) that one of the differences of a stabilized from an unstabilized film is manifested in the origin, because of elasticity of the surfactant layer, of traveling waves whose frequency grows as the elasticity increases, while the damping decrement remains constant.

*The author learned of the existence of [9, 10] after the paper had been sent to press.

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